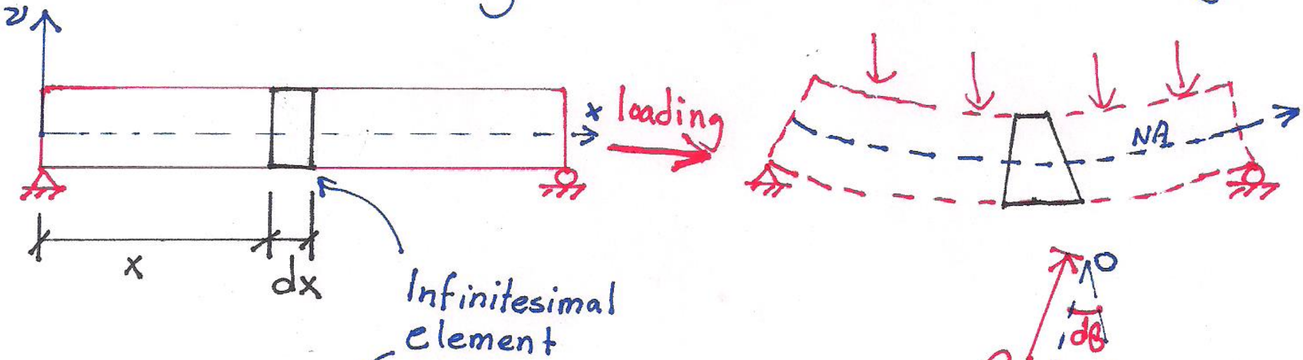
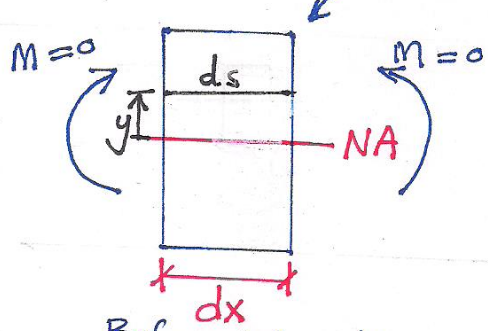
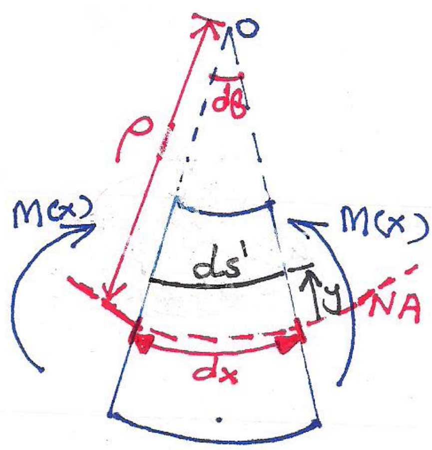
Elastic beam theory and double integration method

Consider the following beam, before and after loading:





Before deformation



After deformation

$$\varepsilon_x(y) = \frac{ds' - ds}{ds} = \frac{(\cancel{\rho}-y)\cancel{d\theta} - \cancel{\rho}\cancel{d\theta}}{\rho\, \cancel{d\theta}} = \frac{-y}{\rho}$$

"The infinitesimal" element is subjected to normal strain (ε)

$$\frac{1}{\rho} = -\frac{\varepsilon_{xx}(y)}{y}$$

For linear-elastic behavior and Homogenous beam:

$\sigma_x = E\,\varepsilon_x$ "Hooke's Law"

$\sigma_x = -\dfrac{M y}{I_z}$ "Flexure" formula

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{\frac{-My}{I_z}}{E} = \frac{-M\cdot y}{E\cdot I_z}$$

$$\frac{1}{\rho} = \cancel{-}\,\frac{\left(\cancel{-}M\cdot\cancel{y}/E\cdot I_z\right)}{\cancel{y}} \Rightarrow \boxed{\frac{1}{\rho} = \frac{M}{EI_z}}$$ curvature

Point "O": is the center of curvature

ρ : is the radius of curvature at the point of the elastic curve

$\frac{1}{\rho}$ = curvature : is the amount by which a curve deviates from being a straight line.

E = material's modulus of elasticity or Young's modulus

I = The beam's moment of inertia about the neutral axis (NA).

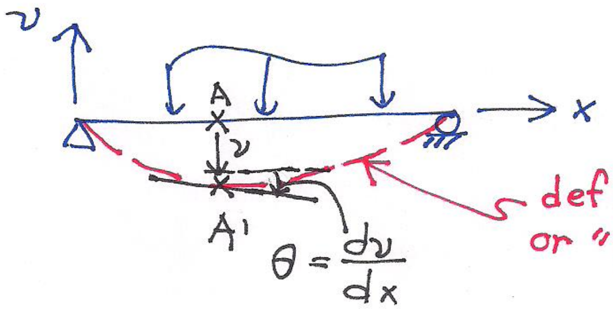
$$\frac{1}{\rho} = \frac{d^2v/dx^2}{\sqrt[2]{\left(1+(dv/dx)^2\right)^3}} = \frac{M}{EI}$$

nonlinear second-order differential equation gives exact shape of the elastic curve assuming that beam deflections occur only due to "bending"

⇛ Shear deformation is ignored

v = vertical deflection of the beam

$\theta = \frac{dv}{dx}$ = slope of the tangent line to the deformed beam (elastic curve)



deflected shape or "Elastic curve" = deflection curve of the longitudinal axis that passes through the centroid of each cross-sectional area of the beam.

For linear-elastic behavior → deformation is very small

$\theta = \frac{dv}{dx} << 1$ $\qquad \theta^2 = \left(\frac{dv}{dx}\right)^2 \approx 0$

$$\boxed{\frac{1}{\rho} = \frac{d^2v}{dx^2}}$$

This makes life much easier.

$$\boxed{\frac{d^2 v}{dx^2} = \frac{M}{EI}}$$

Second order ordinary differential equation.

$d^2v/dx^2 = M/EI$ = curvature $= 1/\rho$ = rate of change in slope

M = internal moment in the beam at the point

EI = flexural rigidity and it is always a positive quantity.

Steps to Calculate deflection by double integration method:

a) Calculate support reactions, if necessary.

b) Write the expression of the internal bending moment $M(x) = ?$

c) Write the boundary and continuity conditions

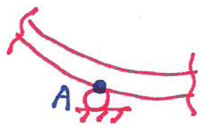
d) $v = \iint \frac{M}{EI}\,dx$, $\theta = \frac{dv}{dx} = \int \frac{M}{EI}\,dx$

E) determine the constants of integration using the conditions specified in part "c".

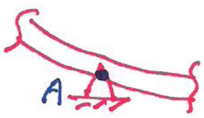
Boundary Conditions

Roller $v_A = 0$ $M_A = 0$

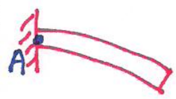
Pin $v_A = 0$ $M_A = 0$



Roller $v_A = 0$



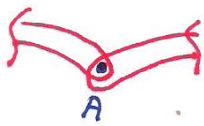
Pin $v_A = 0$



Fixed end $v_A = 0$ $\frac{dv}{dx} = \theta_A = 0$

Free end (left end)

$M_A = 0$ shear$_A = 0$



Internal pin or hinge

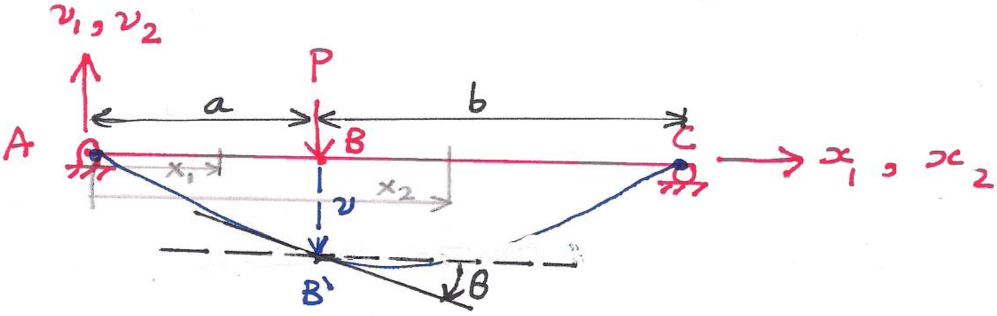
$M_A = 0$

Continuity Conditions

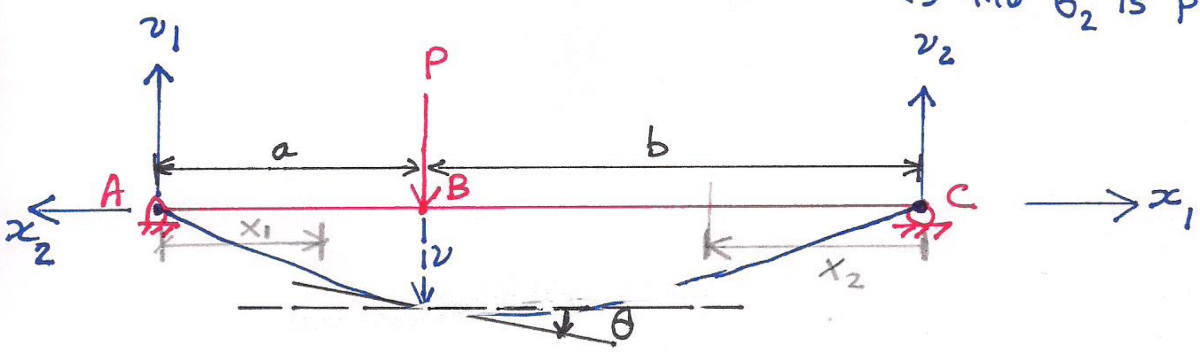
For the beam shown, two x- coordinates are chosen with origins at A. Each is valid only within the regions $0 \leq x_1 \leq a$ and $a \leq x_2 \leq (a+b)$. One the functions for the slope and deflection are obtained, they must give the same values for the slope and deflection at point B so the elastic curve is physically continuous.

$$\theta_1(x_1 = a) = \theta_2(x_2 = a)$$

$$v_1(x_1 = a) = v_2(x_2 = a)$$

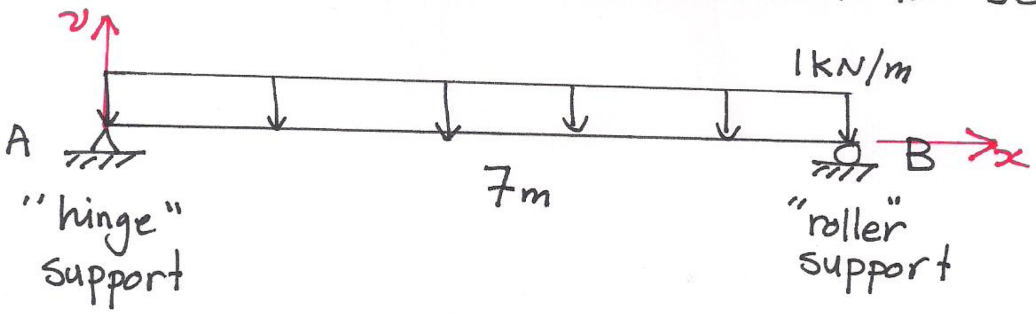


If instead the elastic curve is expressed in terms of the coordinates $0 \leq x_1 \leq a$ and $0 \leq x_2 \leq b$, as shown below, then the conti of slope and deflection at B requires $\theta_1(x_1 = a) = -\theta_2(x_2 = b)$ and $v_1(x_1 = a) = v_2(x_2 = b)$. In this particular case, a negative sign is necessary to match the slopes at B since x_1 extends positive to the right, whereas x_2 extends positive to the left. Consequently, θ_1 is positive counterclockwise, and θ_2 is positive clockwise.



Example

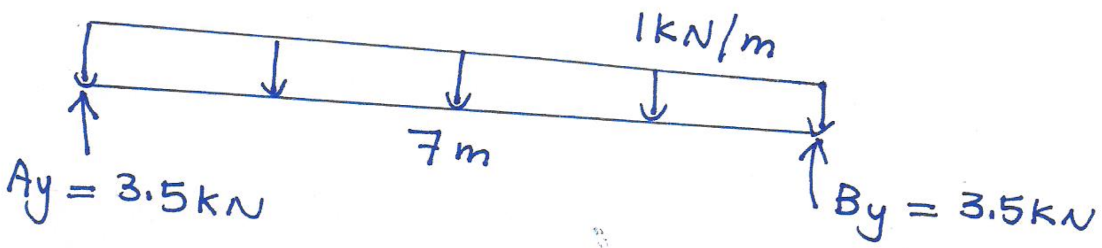
Use the double integration method to determine the location and magnitude of maximum deflection for beam AB.



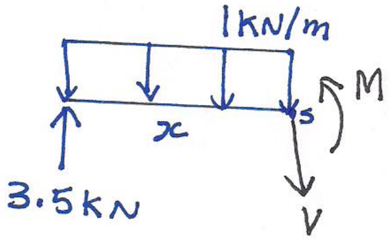
EI = constant

Solution

a) Determine the reactions at A and B:



b) Write the internal moment as a function of x:



$$\circlearrowleft \Sigma M_s = 0 : \quad +M + (1)(x)\left(\frac{x}{2}\right) - 3.5x = 0$$

$$M = -\frac{x^2}{2} + 3.5\,x$$

$$0 \leq x \leq 7\,m$$

note: "M" is represented by single function
⇒ θ, v each will have single function
⇒ No continuity conditions

c) $\frac{d^2 v}{dx^2} = \frac{M}{EI}$

$EI \cdot \frac{d^2 v}{dx^2} = M \rightarrow EI \frac{d^2 v}{dx^2} = -\frac{x^2}{2} + 3.5x$

$EI \int \frac{d^2 v}{dx^2} = \int \left(-\frac{x^2}{2} + 3.5x\right) dx \rightarrow EI \frac{dv}{dx} = -\frac{x^3}{6} + \frac{3.5}{2} x^2 + C_1$

slope

$EI \int \frac{dv}{dx} = \int \left(-\frac{x^3}{6} + \frac{3.5}{2} x^2 + C_1\right) dx$

deflection

$EIv = \frac{-x^4}{24} + \frac{3.5}{6} x^3 + C_1 x + C_2$

$C_1 = ?$ and $C_2 = ?$

d) boundary conditions :

Hinge support at A : deflection = 0 $\quad v_A = 0$ at $x = 0$

Roller support at B : $\quad v_B = 0$ at $x = 7m$

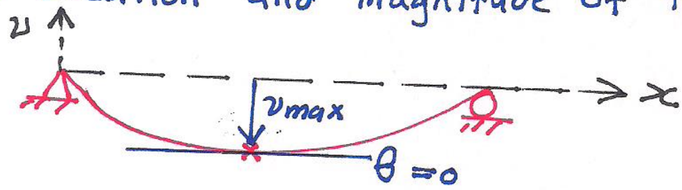
At $x = 0$; $EI(0) = 0 + 0 + 0 + C_2 \rightarrow C_2 = 0$

At $x = 7m$: $EI(0) = \frac{-(7)^4}{24} + \frac{3.5(7)^3}{6} + C_1(7) \rightarrow C_1 = \frac{-343}{24}$

$\frac{dv}{dx} = \frac{1}{EI}\left[-\frac{x^3}{6} + \frac{3.5}{2} x^2 - \frac{343}{24}\right]$ "slope" radian

$v = \frac{1}{EI}\left[\frac{-x^4}{24} + \frac{3.5x^3}{6} - \frac{343}{24} x\right]$ " deflection" meter

e) Location and magnitude of the maximum deflection:



→ The slope is zero at the maximum deflection

$$\frac{dv}{dx} = 0 \;; \quad \frac{-x^3}{6} + \frac{3.5}{2}x^2 - \frac{343}{24} = 0$$

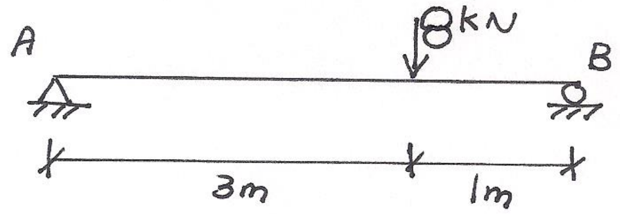
$$x = 3.5\ m$$

$$v_{max} = v\Big|_{x=3.5\,m} = \frac{1}{EI}\left[-\frac{3.5^4}{24} + \frac{3.5(3.5)^3}{6} - \frac{343}{24}(3.5)\right]$$

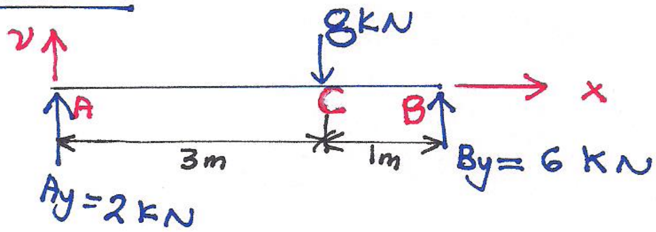
$$= \frac{-12005}{384\,EI}\ m \quad (\downarrow) \qquad\qquad v_{max} = \frac{5wL^4}{384EI}$$

Example

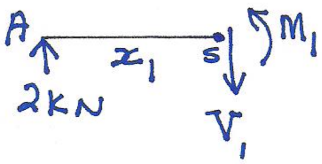
For the beam shown, Calculate the maximum deflection



Solution



Part AC



$\circlearrowleft + \sum M_s = 0$

$+M_1 - 2x_1 = 0$

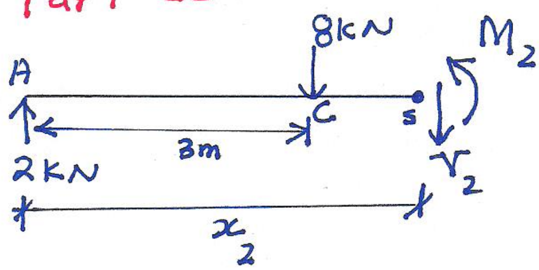
$M_1 = 2x_1$

$0 \leqslant x_1 \leqslant 3$

$EI \dfrac{d^2 v_1}{dx_1^2} = 2x_1 \xrightarrow{\int} EI \dfrac{dv_1}{dx_1} = x_1^2 + C_1$

$\xrightarrow{\int} EI\, v_1 = \dfrac{x_1^3}{3} + C_1 x_1 + C_2$

Part CB



$\circlearrowleft \sum M_s = 0 : +M_2 + 8(x_2 - 3) - 2x_2 = 0$

$M_2 = -6x_2 + 24$

$3m \leqslant x_2 \leqslant 4m$

$EI \dfrac{d^2 v_2}{dx_2^2} = -6x_2 + 24$

$EI \dfrac{dv_2}{dx_2} = -3x_2^2 + 24x_2 + C_3$

$EI\, v_2 = -x_2^3 + 12x_2^2 + C_3 x_2 + C_4$

Boundary Conditions at A and B: $v_1 \Big|_{x_1 = 0} = 0$ $\qquad v_2 \Big|_{x_2 = 4m} = 0$

Continuity Conditions at c: $v_1 \Big|_{x_1 = 3m} = v_2 \Big|_{x_2 = 3m}$

$\dfrac{dv_1}{dx_1}\Big|_{x_1 = 3m} = \dfrac{dv_2}{dx_2}\Big|_{x_2 = 3m}$

$$v_1\Big|_{x_1=0} = 0 \Rightarrow 0 = 0 + 0 + C_2 \rightarrow C_2 = 0$$

$$v_2\Big|_{x_2=4} = 0 \Rightarrow 0 = -(4)^3 + 12(4)^2 + C_3(4) + C_4 \quad \text{—} ①$$

$$v_1\Big|_{x_1=3m} = v_2\Big|_{x_2=3} \Rightarrow \frac{(3)^3}{3} + C_1(3) = -(3)^3 + 12(3)^2 + C_3(3) + C_4 \quad \text{—} ②$$

$$\frac{dv_1}{dx_1}\Big|_{x_1=3m} = \frac{dv_2}{dx_2}\Big|_{x_2=3m} \Rightarrow (3)^2 + C_1 = -3(3)^2 + 24(3) + C_3 \quad \text{—} ③$$

Solve equations 1, 2, and 3 for the constants C_1, C_3, C_4:

$$C_1 = -5 \quad , \quad C_3 = -41 \quad , \quad C_4 = 36$$

$$\frac{dv_1}{dx_1} = \frac{1}{EI}\left[x_1^2 - 5\right]$$

$$0 \leq x_1 \leq 3m$$

$$v_1 = \frac{1}{EI}\left[\frac{x_1^3}{3} + -5x_1\right]$$

$$\frac{dv_2}{dx_2} = \frac{1}{EI}\left[-3x_2^2 + 24x_2 - 41\right]$$

$$3_m \leq x_2 \leq 4m$$

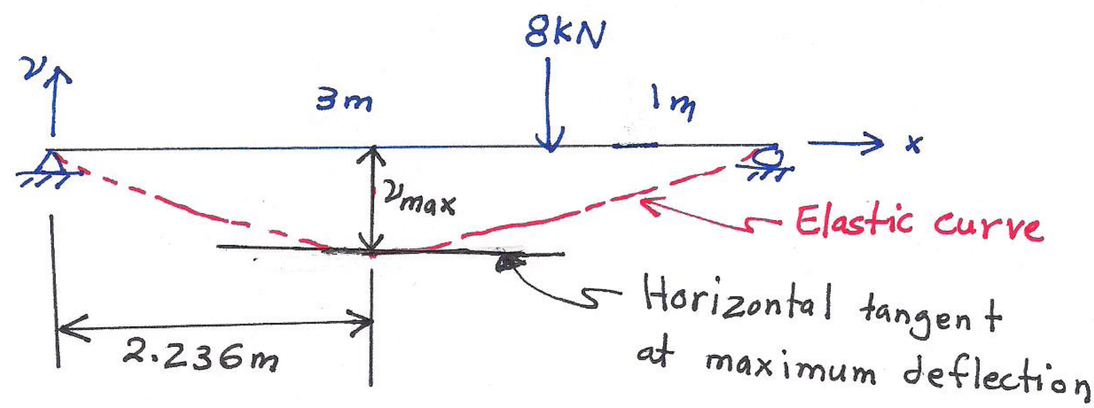
$$v_2 = \frac{1}{EI}\left[-x_2^3 + 12x_2^2 - 41x_2 + 36\right]$$

Assume the maximum deflection is within part AC:

$$\frac{dv_1}{dx_1} = 0 : \qquad x_1^2 - 5 = 0 \rightarrow x_1 = \sqrt{5} = 2.236m < 3m$$

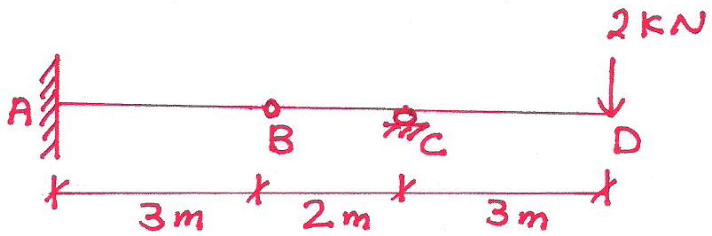
$$v_{max} = v_1\Big|_{x_1 = 2.236m} = \frac{1}{EI}\left[\frac{(2.236)^3}{3} - 5(2.236)\right]$$

$$= \frac{-7.453}{EI} m \;(\downarrow)$$



Example

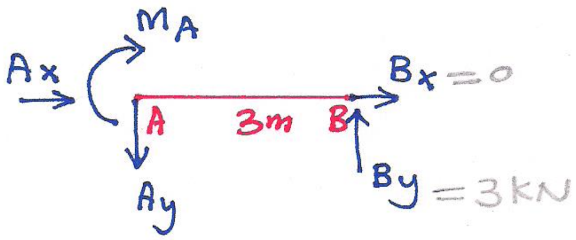
Beam ABCD shown has a fixed support at A, an internal hinge at B, a roller support at C, and a free end at D. Determine the deflection and slope at points B and D.

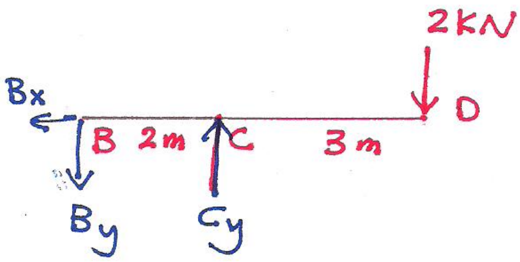


$EI =$ constant

Solution

Determine the reactions at A and C





$$\Sigma F_x = 0 \rightarrow \quad A_x = 0$$

$$\overset{+}{\uparrow} \Sigma F_y = 0: \; -A_y + 3 = 0$$

$$A_y = 3 \text{ kN}$$

$$\overset{+}{\circlearrowleft} \Sigma M_B = 0: -M_A + 3(3) = 0$$

$$M_A = 9 \text{ kN.m}$$

$$\Sigma F_x = 0 \rightarrow B_x = 0$$

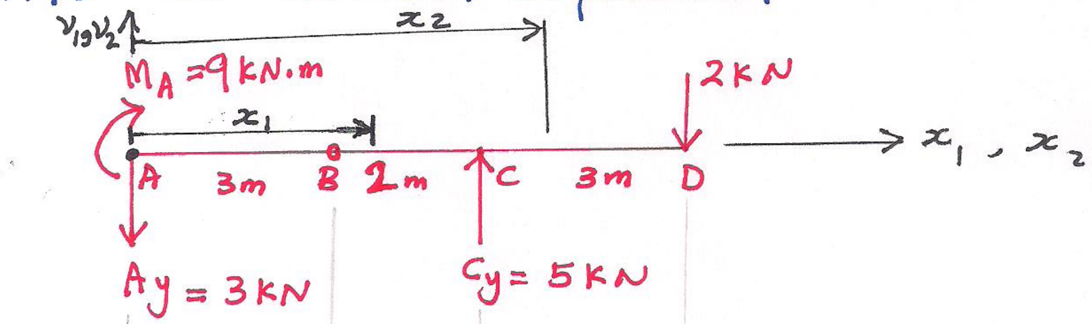
$$\overset{+}{\circlearrowleft} \Sigma M_B = 0: \; -2(5) + 2C_y = 0$$

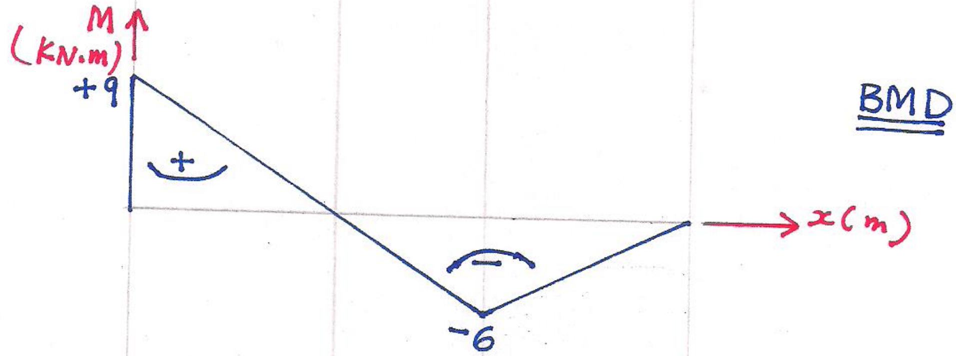
$$C_y = 5 \text{ kN} (\uparrow)$$

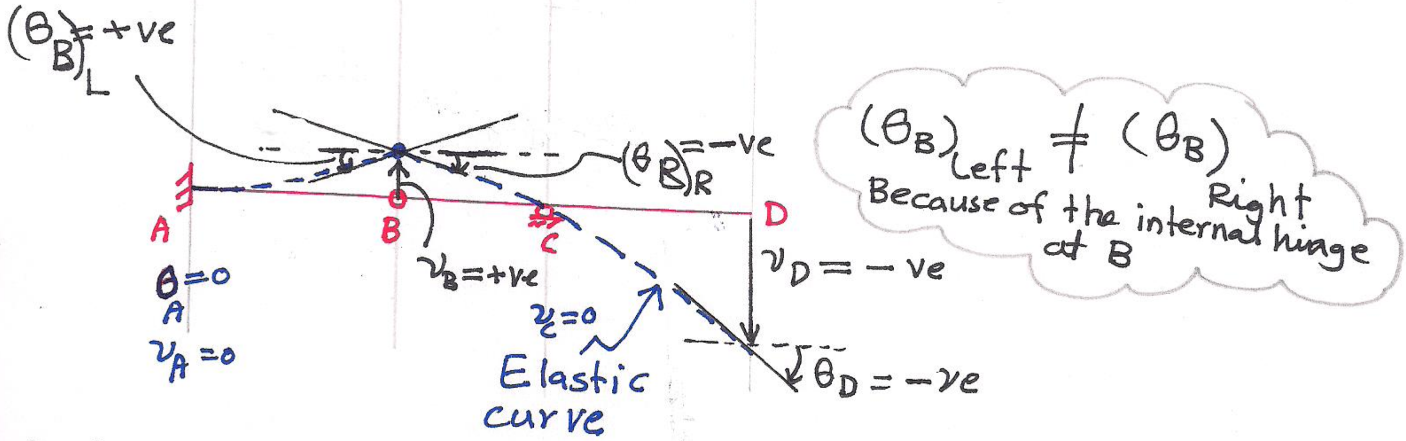
$$\overset{+}{\uparrow} \Sigma F_y = 0: \; -B_y + 5 - 2 = 0$$

$$B_y = 3 \text{ kN} (\downarrow)$$

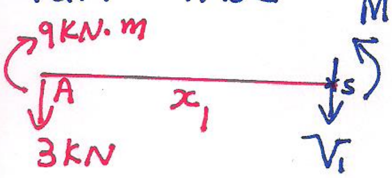
Write the moment expression:







Part ABC

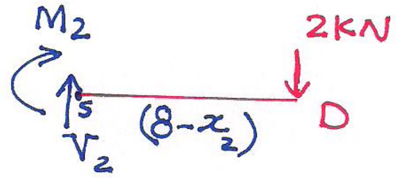


$$\circlearrowleft^{+} \Sigma M_s = 0 : +M_1 + 3x_1 - 9 = 0$$

$$M_1 = 9 - 3x_1$$

$$0 \leq x_1 \leq 5\,m$$

Part CD



$$\circlearrowleft^{+} \Sigma M_s = 0 : -M_2 - 2(8-x_2) = 0$$

$$M_2 = -16 + 2x_2$$

$$5\,m \leq x_2 \leq 8\,m$$

Part AB:

$0 \leqslant x_1 \leqslant 3\,m$

$$EI\frac{d^2 v_{AB}}{dx_1^2} = 9 - 3x_1 \xrightarrow{\int} EI\frac{dv_{AB}}{dx_1} = 9x_1 - \frac{3}{2}x_1^2 + C_1$$

$$\downarrow \int$$

$$EI\, v_{AB} = \frac{9}{2}x_1^2 - \frac{1}{2}x_1^3 + C_1 x_1 + C_2$$

$$v_{AB}\Big|_{x_1=0} = 0 : \quad 0 = 0 - 0 + 0 + C_2 \rightarrow C_2$$

$$dv_{AB}/dx_1\Big|_{x_1=0} = 0 : \quad 0 = 0 - 0 + C_1 \rightarrow C_1 = 0$$

$$\frac{dv_{AB}}{dx_1} = \frac{1}{EI}\left[9x_1 - \frac{3}{2}x_1^2\right] \qquad 0 \leqslant x_1 \leqslant 3\,m$$

$$v_{AB} = \frac{1}{EI}\left[\frac{9}{2}x_1^2 - \frac{1}{2}x_1^3\right]$$

not to 5 m
⇒ There is a discontinuity at B because of the internal hinge.

Part BC

$3\,m \leqslant x_1 \leqslant 5\,m$

$$EI\frac{d^2 v_{BC}}{dx_1^2} = 9 - 3x_1 \xrightarrow{\int} EI\frac{dv_{BC}}{dx_1} = 9x_1 - \frac{3}{2}x_1^2 + C_3$$

$$\downarrow \int$$

$$EI\, v_{BC} = \frac{9}{2}x_1^2 - \frac{x_1^3}{2} + C_3 x_1 + C_4$$

$$v_{BC}\Big|_{x_1=5m} = 0 : \quad 0 = \frac{9}{2}(5)^2 - \frac{(5)^3}{2} + C_3(5) + C_4$$

$$5C_3 + C_4 = -50 \quad \text{—} (1)$$

$$v_{AB}\Big|_{x_1=3\,m} = v_{BC}\Big|_{x_1=3\,m}$$

$$\frac{1}{EI}\left[\frac{9}{2}(3)^2 - \frac{1}{2}(3)^3\right] = \frac{1}{EI}\left[\frac{9}{2}(3)^2 - \frac{(3)^3}{2} + C_3(3) + C_4\right]$$

$$3C_3 + C_4 = 0 \quad \text{—— (2)}$$

From equations (1) and (2): $C_3 = -25$

$C_4 = 75$

$$\frac{dv_{BC}}{dx_1} = \frac{1}{EI}\left[9x_1 - \frac{3}{2}x_1^2 - 25\right] \quad 3m \leq x_1 \leq 5m$$

$$v_{BC} = \frac{1}{EI}\left[\frac{9}{2}x_1^2 - \frac{x_1^3}{2} - 25x_1 + 75\right]$$

Part CD $5m \leq x_2 \leq 8\,m$

$$EI\frac{d^2v_{CD}}{dx_2^2} = -16 + 2x_2 \xrightarrow{\int} EI\frac{dv_{CD}}{dx_2} = -16x_2 + x_2^2 + C_5$$

$$\Big\downarrow \int$$

$$EI\, v_{CD} = -8x_2^2 + \frac{x_2^3}{3} + C_5 x_2 + C_6$$

$$v_{CD}\Big|_{x_2=5\,m} = 0 \;:\; 0 = -8(5)^2 + \frac{(5)^3}{3} + 5C_5 + C_6$$

$$5C_5 + C_6 = \frac{475}{3} \quad \text{—— (3)}$$

$$\left.\frac{dv_{BC}}{dx_1}\right|_{x_1=5\,m} = \left.\frac{dv_{CD}}{dx_2}\right|_{x_2=5\,m}$$

$$\frac{1}{EI}\left[9(5)-\frac{3(5)^2}{2}-25\right] = \frac{1}{EI}\left[-16(5)+(5)^2+C_5\right]$$

$$C_5 = 37.5$$

Use equation ③ to find C_6: $\quad C_6 = -\dfrac{175}{6}$

$$\frac{dv_{CD}}{dx_2} = \frac{1}{EI}\left[-16x_2+x_2^2+37.5\right] \qquad 5_m \leq x_2 \leq 8m$$

$$v_{CD} = \frac{1}{EI}\left[-8x_2^2+\frac{x_2^3}{3}+37.5x_2-\frac{175}{6}\right]$$

$$(\theta_B)_L = \left.\frac{dv_{AB}}{dx_1}\right|_{x_1=3m} = \frac{1}{EI}\left(9(3)-\frac{3}{2}(3)^2\right) = \frac{+27}{2EI}\,\text{rad}\ (\circlearrowleft) \quad \theta_L$$

$$(\theta_B)_R = \left.\frac{dv_{BC}}{dx_1}\right|_{x_1=3m} = \frac{1}{EI}\left[9(3)-\frac{3}{2}(3)^2-25\right] = \frac{-23}{2EI}\,\text{rad}\ (\circlearrowright) \quad \theta_R$$

$$v_B = v_{AB}\Big|_{x_1=3m} = v_{BC}\Big|_{x_1=3m} = \frac{1}{EI}\left[\frac{9}{2}(3)^2-\frac{1}{2}(3)^3\right] = \frac{+27}{EI}\,m\ (\uparrow)$$

$$\text{or}\ \frac{1}{EI}\left[\frac{9}{2}(3)^2-\frac{1}{2}(3)^3-25(3)+75\right] = \frac{+27}{EI}\,m$$

$$\theta_D = \left.\frac{dv_{CD}}{dx_2}\right|_{x_2=8m} = \frac{1}{EI}\left[-16(8)+(8)^2+37.5\right] = \frac{-53}{2EI}\,\text{rad} \quad \theta_D$$

$$v_D = v_{CD}\Big|_{x_2=8m} = \frac{1}{EI}\left[-8(8)^2+\frac{(8)^3}{3}+37.5(8)-\frac{175}{6}\right] = -\frac{141}{2EI}\,m\ (\downarrow)$$

The maximum deflection is at point D which the free end